Worksheet for Sections 6.7, 6.2, and 6.8

1. Consider the curves C_1 and C_2 parametrized by $x = \cos t, y = \sin t$, and $x = \sin 2t, y = \cos 2t$, respectively, for $0 \le t \le 2\pi$. Draw the graphs of C_1 and C_2, and indicate all values of t in $[0, 2\pi]$, if any, for which $C_1(t) = C_2(t)$.

2. (a) Under what conditions is it possible to consider a parametrized curve as the graph of a function? (*Hint*: If $x = f(t)$ and $y = g(t)$ for $a \le t \le b$, what might one let $f(t)$ be?)

   (b) Write down parametric equations for a general line l.

   (c) We are given a differentiable function f on $[a, b]$, whose graph is C. Derive the formula for the length L of C from the formula for the length of C when f is given parametrically.

3. Let $f(x) = \sin x$, for $0 \le x \le \pi/2$.

   (a) Write down a formula for the length L of the graph of f.

   (b) By using the Comparison Property for integrals, show that $\pi/2 \le L \le \pi$.

4. (a) Let $f(t) = t - \sin t$ for all t. Show that f is a strictly increasing function of t, and determine the inflection points of the graph of f.

   (b) Consider the cycloid C parametrized by $x = t - \sin t$ and $y = 1 - \cos t$, for all real t. Find $C(0), C(\pi)$, and $C(2\pi)$, and show that the highest point on C occurs for $t = \pi$. Tell why this shows that one arch of the cycloid is *not* a semicircle.

   (c) Let $P(t) = (t - \sin t, 1 - \cos t)$ for all real t. Use (a) to show that if $t_1 \ne t_2$, then $x(t_1) \ne x(t_2)$. (This means that the cycloid C is the graph of a function!)

5. This problem is taken from Project 1 on p. 383, which is related to Archimedes' approximation of the circumference of a circle by finding lengths of inscribed (regular) polygons such as those in Figure 6.26. Here we will assume that the point (1, 0) is one vertex of each of the inscribed polygons.

   (a) Consider the inscribed regular polygon of n sides. Show that a vertex adjacent to (1, 0) is $(\cos(2\pi/n), \sin(2\pi/n))$.

   (b) For the inscribed polygon of n sides in (a), show that the length of the side S determined by the 2 points in (a) is $\sqrt{2 - 2\cos(2\pi/n)}$, and show that expression equals $2\sin(\pi/n)$.

   (c) Using the result of (b), find the length L_n of the inscribed regular polygon of n sides.

   (d) By letting n be large and using a limit involving $(\sin t)/t$, show why the limit L of L_n as n increases without bound should be 2π.

6. Let $r > 0$. The equations $x = r\cos^3 t$ and $y = r\sin^3 t$ parametrize an astroid (Figure 6.84 in the book). Draw the graph and find the length L of the astroid. (*Hint*: Square roots are nonnegative!)